Asymptotic Study of Divorce Model with Pre-Marriage Preparedness as Control

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Abstract

Peaceful cohabitation in a marriage institution is challenged with separation/divorce because of distinct individual psychological build-up. A deterministic model for the divorce epidemic was proposed using standard incidence as a forcing function. The stability theory of differential equations was used to perform the model analysis qualitatively on which the equilibria obtained are locally and globally stable. Bifurcation and sensitivity analysis of the model were performed; parameters responsible for managing and eradicating the spread of divorce in marriages were determined. A numerical simulation was performed with results that showed pre-marriage preparedness and conscientious growth in tolerance of individual differences as a stabilizer to marriages.

Keywords: mathematical modeling; divorce; equilibria; reproduction number; sensitivity analysis; bifurcation.

1   Introduction

Marriage can be viewed differently within cultures, religions, environments, countries, personal factors, and seasons. It can be termed as a socially or ritually recognized union if all the rites where performed, which establishes rights and obligations between the spouses, as well as their children both biologically or adopted, and relatives [14]. Succinctly defined, marriage in Nigeria is a socially recognized and approved (customary, religious, statutory, cultural, or traditional) union of man and woman who dedicate to each other with the sole aspiration of a permanent and long time relationship [18] with a legal obligation to each other throughout their lives. Regardless of the conjugal bliss, heinous experiences have dripped into the unions whose effects are divorce and separation. Divorce is the canceling, dissolution, or reorganizing of the legal duties and responsibilities attached to marriage under the rule of law that establish such union. Separation, on the other hand is the abrogation of cohabitation in marriage. In 2018, 70 % of women in Nigeria aged 15 to 49 years [8] were married, which portrays the essence placed on marriage in Nigeria. Out of the total population of Nigeria in 2020, over 39,149,232 were married, 357,508 were divorced and 1,057,236 were separated which 75% of the separated couples end up in divorce [3].

Divorce is an endemic issue that seriously affects the social and economic structure of contemporary society as much as any disease because of its adverse effect on personal stability and children. Before 2016, divorce was uncommon in Nigeria with statistics of about 0.2% of men and 0.3% of women that have legally dissolved their marriage. Furthermore, in 2018, it was reported Vanguard Newspaper that in Nigeria rate of divorce/separation increased by 14% which showed its ugly head in over 3,000 divorce cases recorded in Badadry, Lagos alone. In a Newspaper article published on 28th February 2020, it was learned that Federal Capital Development Authority (FCDA) receives 20 to 30 cases of divorce issues every working day, which corresponds to a claim of the High court in Federal Capital Territory (FCT) that between ending of 2019 and February 2020, over 2000 divorce cases were filed which averages 30 cases being entertained daily. The prevalence of divorce has raised a great concern for the children whose families are broken especially at a tender age. A great number of these children are associated with illicit substances, social withdrawal, poor academic performance, prone to teen pregnancy, the victim of adult mental problems, and exhibit externalizing and internalizing behavioural problems [15]. The causes of divorce and separation in marriages are ranked but are not limited to lack of commitment, unchecked arguments, infidelity, early marriage, unwanted and unrealistic expectations, lack of equality in a relationship, violence, etc. Summarily, divorce has enormous societal, cultural, economical, political, psychological, and all-encompassing consequences [1, 10], hence for a stable, wealthy, peaceful, and productive society the family/marriage of a given society should be ultimately restored to its original purpose.

For the past decades, mathematical modelling has been employed in studying epidemics [23, 16], human physiology, the flow of fluids, agriculture, structural development, pigmentation, mutation, pneumonia/Typhoid [22], childhood disease [5], railway system [2] and students performance in Mathematics examination [11] etc. We will consider divorce as an epidemic in contemporary society because of its apocalyptic effect on family life which is the sole and integral unit of the society. Different researchers have worked on models related to divorce; a model that studied disparity in marital satisfaction, as well as the economy, was studied in [7]. They concluded that harmonization and limitation of social contagion will reduce the rate of divorce. A model of three compartments; married, divorced, and separated was formulated by [9] which was extended by [20]. In their study, they concluded that increasing the number of marriages that go into separation and educating them along with reducing contact with infectious divorcees help in combating the divorce epidemic. A study on qualitative analysis of a mathematical model of divorce epidemic with anti-divorce therapy was propounded by [13], they found out that anti-divorce protocols and

reconciliation can jointly stabilize marriages. Bifurcation analysis of mathematical model analysis of marriage divorce [21] was performed with the conclusion that divorce has eaten deep into the fiber of the society. We construct a mathematical model for the dynamics of divorce in the society incorporating personal preparation for marriage on the part of singles who are ready for marriage and personal development on tolerance of negative affects of our spouse as effective control measures, which are features missing in other models. The standard incidence rate is used as the force of infection with marriage disorder resulting as a result of contact (virtually or physically) with divorcees and some unrepentant separated individuals who will eventually get a divorce which is also missing in other works. The model proposed is analyzed both asymptotically and numerically to understand the intricacies and effects of the controls on divorce dynamics in Nigeria.

The paper is organized as follows: In Section 2, we formulate the model and study the properties of the model. We perform the model analysis which involves obtaining the equilibria, effective reproduction number, and study of the global stability of divorce-free equilibrium using the Lyapunov method, the local and global stability of divorce endemic equilibrium is explored and the bifurcation analysis in Section 3. Sensitivity analysis of the parameter in effective reproduction number is performed in Section 4. The numerical simulation and results are performed in Section 5 while laconic conclusions are presented in Section 6.

2   Materials and Methods

We propose a mathematical model to analyze the dynamics of divorce in Nigeria where married persons are exposed to the divorce epidemic by interaction with divorcees or unperturbed separated individuals through contact, which is either by physical, virtual, or hybrid capacity. The population is divided into six compartments; S represents the singles who are of marriageable age and are contextually ready for marriage. Marriages are categorized based on marital interaction, satisfaction, and ability to manage marital and social feedback. Marriage with no negative affects is an illusion, hence there is a constriction of the natural flow of emotions every day; what differentiates marriages is the psychology applied to the union. H represents unstable marriage which includes hostile and hostile - detached marriages characterized by the four-horse of the apocalypse (Criticism, contempt, defensiveness, and stonewalling), and escalation of negative affects [12]. M represents stable marriage which includes validators, volatiles, and avoiders marriages. They differ in the amount and timing for persuasions, and are characterized by their ability to consider negativity as fleeting, situational and positivity [12]. The population of marriages undergoing counselling or separated is denoted by C while the divorced or legally dissolved marriage is represented as D while R is the population of the restored marriages after a series of reconciliation.

Λ is the influx of singles who are ready for marriage, they get married at the rate k. The level of preparedness for the course of a marriage is denoted as r, which is key to this model because marriage is a lifetime project, hence singles need to be equipped with all available information that will make marriage work by psychological reorientation. The couples interact with the environment in which they come in contact with both the divorced and separated, where seeds are sowed by disguised sharing of experiences. ω represents the level of tolerance of negative affect emanating from marriage institution and society; β is the divorce rate and τ is the rate of adapting and attuning to divorce by the separated population. Some of the stable marriages are overwhelmed by the infective contact with the divorced and unperturbed separated individuals and enter into counselling/separated class at the rate α. The proportion of unstable marriage that results in divorce or separation after the infective contact is ρ which may be cautioned by the level of dysfunctionality,

ψ in the marriage. Some of the separated marriages got a permanent divorce at the rate h, the divorced join the single sub-population at the rate b when they are ready to remarry. The unstable marriage is filled with rancour, abuse, and violence of all sorts, there is a tendency that it may lead to death at the rate d_1 and the divorcees sometimes pass through emotional and psychological blackouts which tend to result in death at the rate of d_2. The rate of restoring separated marriages through reconciliation is δ and γ is the rate of restoring divorced marriages through reconciliation situated by the society. All the sub-population have natural death at the rate μ. We assume homogeneous interaction between all parties and the probability of being divorced is not affect by social status, tribe, sex, and age. The schematic diagram for the model is shown in Figure 1.
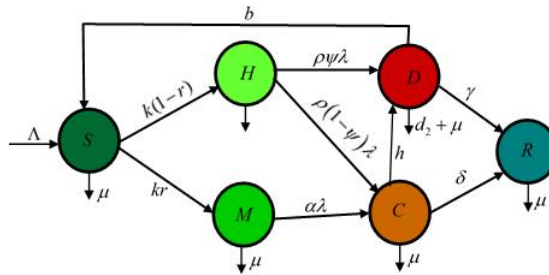

Figure 1: Schematic diagram of the proposed model.

In context of the above assumptions, the model is governed by the following system of differential equations:

$$
\left.
\begin{array}{rcl}
\frac{dS}{dt} &=& \Lambda + bD - (k+\mu)S, \\
\frac{dH}{dt} &=& k(1-r)S - (\rho\lambda + d_1 + \mu)H, \\
\frac{dM}{dt} &=& krS - (\alpha\lambda + \mu)M, \\
\frac{dC}{dt} &=& \alpha\lambda M + \rho(1-\psi)\lambda H - (h+\delta+\mu)C, \\
\frac{dD}{dt} &=& \rho\psi\lambda H + hC - (b+\gamma+d_2+\mu)D, \\
\frac{dR}{dt} &=& \gamma D + \delta C - \mu R,
\end{array}
\right\}
\tag{1}
$$

where $\lambda = \dfrac{(1-\omega)\beta(\tau C + D)}{N}$, $N(t) = S(t) + H(t) + M(t) + C(t) + D(t) + R(t)$ with feasible domain

$$
\Omega = \left\{ S(0) > 0, H(0) \ge 0, M(0) \ge 0, C(0) \ge 0, D(0) \ge 0, R(0) \ge 0 | N \le \frac{\Lambda}{\mu} \right\}.
\tag{2}
$$

The biological, sociological and psychological meaning of all the parameters in (1) are given in Table 1, they all assume non-negative numerical values.

3   Model Analysis

3.1   Invariant region

Theorem 3.1. *The closed region* $\Omega = \left\{ (S, H, M, C, D, R) \in \mathbb{R}_+^6 : 0 < N \le \frac{\Lambda}{\mu} \right\}$ *is positively invariant set for model* (1).

Table 1: Biological, sociological and psychological interpretation of parameters involved in the model system 1.

Parameters	Description
Λ	Influx of singles who are ready for marriage
b	Rate at which the divorced become ready to remarry
k	Rate of marriage
μ	Natural death rate
r	Marriage preparedness by the singles
ρ	Proportion of unstable marriage that leads to either separation /divorce
α	Proportion of couples that divorce after separation
ψ	Level of dysfunctional and escalated negative affect
h	Proportion of couples that divorce after separation
δ	Rate of restoring separated marriage through reconciliation
γ	Rate of restoring divorced couples through reconciliation
ω	Level of tolerance of negative affects
β	Divorce rate
τ	Modification factor for separated couples
t	Time in years
State Variables	**Description**
$S(t)$	Number of singles who are of marriageable age and are ready for marriage at t.
$H(t)$	Population of unstable marriages at t
$M(t)$	Population of stable marriages at t
$C(t)$	Population of separated couples at t
$D(t)$	Population of divorced couples at t
$R(t)$	Population of restored marriages at t

Proof. Since $N = S + M + H + C + D + R$ then $\frac{dN}{dt} = \Lambda - d_1 H - d_2 D - \mu N \leq \Lambda - \mu N$. It follows that

$$N(t) \leq \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu} \right) e^{-\mu t}.$$

If $t \to \infty$, we have that $\limsup\limits_{t \to \infty} N(t) \leq \frac{\Lambda}{\mu}$. Hence $\Omega = \left\{ (S, H, M, C, D, R) \in \mathbb{R}_+^6 : 0 < N \leq \frac{\Lambda}{\mu} \right\}$ is the feasible solution set for the model (1) and all solutions of the model are bounded for all $t \in [0, \infty)$. $\qquad\square$

3.2 Positivity of solutions

In this section, we show by means of a theorem that all the solution of the model (1) remain positive for time $t \in [0, \infty)$ if (2) holds.

Theorem 3.2. *All solutions of model* (1) *remains non - negative for all* $t \in [0, \infty)$ *when* (2) *is satisfied.*

Proof. Let $\mathfrak{L} = \sup \left\{ t > 0 : S(\bar{t}) \geq 0, H(\bar{t}) \geq 0, M(\bar{t}) \geq 0, D(\bar{t}) \geq 0, C(\bar{t}) \geq 0, R(\bar{t}) \geq 0 \,\forall\, \bar{t} \in [0, t] \right\}$. Observe that $\mathfrak{L} > 0$ since $S(t) > 0, H(t) \geq 0, M(t) \geq 0, D(t) \geq 0, C(t) \geq 0$ and $R(t) \geq 0$ and when $\mathfrak{L} < \infty$, then either one of $S(t), H(t), M(t), D(t), C(t)$ or $R(t)$ is equal to zero at $\mathfrak{L}$. From (1) we have that

$$\frac{dS}{dt} + (k + \mu)S = \Lambda + bD.$$

Applying variation of constants for the solution at $\mathfrak{L}$ we obtain

$$S(\mathfrak{L}) = S(0) \exp\left(-\int_0^{\mathfrak{L}} (\Lambda + b)(s)ds\right) + \int_0^{\mathfrak{L}} (\Lambda + bD).\exp\left(-\int_s^{\mathfrak{L}} (\Lambda + b)(\bar{t})d\bar{t}\right) ds.$$

Since all variables are positive in $[0, \bar{t}]$, we have that $S(\mathfrak{L}) > 0$.

In the other hand, by approaching other variables in model (1) with the same technique, it is easy to show that

$$
\begin{aligned}
H'(t) &\geq -(\rho\lambda + d_1 + \mu) \Rightarrow H(t) \geq 0, \\
M'(t) &\geq -(\alpha\lambda + \mu) \Rightarrow M(t) \geq 0, \\
C'(t) &\geq -(h + \delta + \mu) \Rightarrow C(t) \geq 0, \\
D'(t) &\geq -(b + \gamma + d_2 + \mu) \Rightarrow D(t) \geq 0, \\
R'(t) &\geq -\mu \Rightarrow R(t) \geq 0.
\end{aligned}
$$

Hence any solution of model (1) when (2) holds is non - negative for $t \in [0, \infty)$. $\qquad\square$

3.3 Equilibrium points

At the steady state, model (1) becomes

$$
\left.
\begin{aligned}
0 &= \Lambda + \mathbf{b}D - \pi_1 S, \\
0 &= k(1-r)S - (\rho\lambda + \pi_2)H, \\
0 &= krS - (\alpha\lambda + \mu)M, \\
0 &= \alpha\lambda M + \rho(1-\psi)\lambda H - \pi_3 C, \\
0 &= \rho\psi\lambda H + hC - \pi_4 D, \\
0 &= \gamma D + \delta C - \mu R,
\end{aligned}
\right\}
\tag{3}
$$

where $\pi_1 = k + \mu, \pi_2 = d_1 + \mu, \pi_3 = h + \delta + \mu$ and $\pi_4 = b + \gamma + d_2 + \mu$. The solution of the system (3) gives the equilibrium state in terms of λ, which after slight simplification we arrived at

$$
\begin{aligned}
S &= \frac{\Lambda\pi_3\pi_4(\rho\lambda+\pi_2)(\alpha\lambda+\mu)}{(\rho\lambda+\pi_2)\left(\pi_1\pi_3\pi_4(\alpha\lambda+\mu)-h\alpha bkr\lambda\right)-\rho bk\lambda(1-r)(\alpha\lambda+\mu)\left(h+\psi(\delta+\mu)\right)}, \\
H &= \frac{\Lambda k\pi_3\pi_4(1-r)(\alpha\lambda+\mu)}{(\rho\lambda+\pi_2)\left(\pi_1\pi_3\pi_4(\alpha\lambda+\mu)-h\alpha bkr\lambda\right)-\rho bk\lambda(1-r)(\alpha\lambda+\mu)\left(h+\psi(\delta+\mu)\right)}, \\
M &= \frac{\Lambda kr\pi_3\pi_4(\rho\lambda+\pi_2)}{(\rho\lambda+\pi_2)\left(\pi_1\pi_3\pi_4(\alpha\lambda+\mu)-h\alpha bkr\lambda\right)-\rho bk\lambda(1-r)(\alpha\lambda+\mu)\left(h+\psi(\delta+\mu)\right)}, \\
C &= \frac{\Lambda\lambda k\pi_4\left(\alpha r(\rho\lambda+\pi_2)+\rho(1-\psi)(1-r)(\alpha\lambda+\mu)\right)}{(\rho\lambda+\pi_2)\left(\pi_1\pi_3\pi_4(\alpha\lambda+\mu)-h\alpha bkr\lambda\right)-\rho bk\lambda(1-r)(\alpha\lambda+\mu)\left(h+\psi(\delta+\mu)\right)}, \\
D &= \frac{\Lambda k\lambda\left(\rho(1-r)(\alpha\lambda+\mu)(h+\psi(\delta+\mu))+h\alpha r(\rho\lambda+\pi_2)\right)}{(\rho\lambda+\pi_2)\left(\pi_1\pi_3\pi_4(\alpha\lambda+\mu)-h\alpha bkr\lambda\right)-\rho bk\lambda(1-r)(\alpha\lambda+\mu)\left(h+\psi(\delta+\mu)\right)}, \\
R &= \frac{\Lambda k\lambda\left(\gamma\{\rho(1-r)(\alpha\lambda+\mu)(h+\psi(\delta+\mu))+h\alpha r(\rho\lambda+\pi_2)\}+\delta\pi_4\{\alpha r(\rho\lambda+\pi_2)+\rho(1-\psi)(1-r)(\alpha\lambda+\mu)\}\right)}{(\rho\lambda+\pi_2)\left(\pi_1\pi_3\pi_4(\alpha\lambda+\mu)-h\alpha bkr\lambda\right)-\rho bk\lambda(1-r)(\alpha\lambda+\mu)\left(h+\psi(\delta+\mu)\right)}.
\end{aligned}
\tag{4}
$$

Model (1) exhibits two steady states:

(a) The divorce endemic equilibrium (DEE) are in two fold E_1 and E_2 are states that divorce are rampant in the society. If $S = S^*, H = H^*, M = M^*, C = C^*, D = D^*, R = R^*, \lambda = \lambda^*$ and $N = N^*$ are as shown in (4) we have that $E_1 = (S^*, H^*, M^*, C^*, D^*, R^*)$ which exists for $(\rho\lambda + \pi_2)(\pi_1\pi_3\pi_4(\alpha\lambda + \mu) - h\alpha bkr\lambda) > \rho bk\lambda(1-r)(\alpha\lambda + \mu)(h + \psi(\delta + \mu))$; and $E_2 = (S^*, H^*, M^*, C^*, D^*, R^*)$ exists for $\pi_1\pi_3\pi_4(\rho\lambda + \pi_2)(\alpha\lambda + \mu) > \rho bk\psi\lambda(1-r)(\alpha\lambda + \mu)(\delta + \mu)$.

(b) The divorce free equilibrium point (DFEP), E_0 is a state that divorce statistics is negligible in the society. It is obtained when the divorce force of infection $\lambda \simeq 0$ which is significant for evaluating DFEP. Then

$$E_0 = (S_0, H_0, M_0, C_0, D_0, R_0) = \left(\frac{\Lambda}{\pi_1}, \frac{k\Lambda(1-r)}{\pi_1\pi_2}, \frac{kr\Lambda}{\mu\pi_1}, 0, 0, 0\right). \tag{5}$$

3.4 Effective reproduction number, R_{eff}

To study the stability of system (1) at divorce free equilibrium we compute a threshold condition called the reproduction number which is the expected number of secondary divorce infections produced by an index case in a completely susceptible population by a typical infective divorcee. To obtain R_{eff}, we make use of next generation matrix as in [24], in which $R_{eff} = \rho(FV^{-1})$ where F is the influx of divorcee related ideology into the compartments, V is the reflux in the divorce prone compartment and ρ is the spectral radius. From the infective compartments we define

$$f_i = \begin{pmatrix} \alpha\lambda M + \rho(1-\psi)\lambda H \\ \rho\psi\lambda H \end{pmatrix} \text{ and } v_i = \begin{pmatrix} \pi_3 C \\ -hC + \pi_4 D \end{pmatrix},$$

then $F = \begin{pmatrix} \dfrac{\beta\tau k(1-\omega)\big(\alpha r\pi_2 + \rho\mu(1-\psi)(1-r)\big)}{\pi_2\mu + k\mu(1-r) + kr\pi_2} & \dfrac{\beta k(1-\omega)\big(\alpha r\pi_2 + \rho\mu(1-\psi)(1-r)\big)}{\pi_2\mu + k\mu(1-r) + kr\pi_2} \\ \dfrac{k\beta\tau\mu\psi(1-\omega)(1-r)}{\pi_2\mu + k\mu(1-r) + kr\pi_2} & \dfrac{k\beta\mu\psi(1-\omega)(1-r)}{\pi_2\mu + k\mu(1-r) + kr\pi_2} \end{pmatrix}$

and $V = \begin{pmatrix} \pi_3 & 0 \\ -h & \pi_4 \end{pmatrix}$.

Hence,

$$R_{eff} = \frac{\beta k(1-\omega)}{\pi_3\pi_4}\left(\frac{\Big(\alpha r\pi_2 + \rho\mu(1-\psi)(1-r)\Big)(\tau\pi_4 + h) + \rho\psi\mu(1-r)}{\pi_2(\mu + kr) + k\mu(1-r)}\right).$$

From theorem 2 in [24], it is established that DFEP is locally asymptotically stable if $R_{eff} < 1$ and unstable when $R_{eff} > 1$.

3.5 Global stability of DFEP

Theorem 3.3. *If $R_{eff} \leq 1$, the DFEP of model (1) is globally asymptotically stable in its feasible region* Ω.

Proof. We establish a Lyapunov function using matrix theoretic approach. Let

$$K(t) = \frac{\beta}{\pi_4} D(t) + \left(\frac{\beta h}{\pi_3 \pi_4} + \frac{\beta \tau}{\pi_3} \right) C(t).$$

Since the parameters are all positive and $D(t)$, $C(t)$ obeys (2), $K(t)$ is positive definite. Therefore

$$
\begin{aligned}
K'(t) &= \frac{\beta}{\pi_4} D' + \left(\frac{\beta h}{\pi_3 \pi_4} + \frac{\beta \tau}{\pi_3} \right) C', \\
&= \frac{\beta}{\pi_4} \left(\rho \psi \lambda H + hC - \pi_4 D \right) + \left(\frac{\beta h}{\pi_3 \pi_4} + \frac{\beta \tau}{\pi_3} \right) \left(\alpha \lambda M + \rho(1-\psi)\lambda H - \pi_3 C \right).
\end{aligned}
$$

After some simple simplification, we have

$$
\begin{aligned}
K'(t) &= \beta(\tau C + D) \left[\beta \rho \psi (1-\omega) \frac{H}{N} + \beta(1-\omega)\left(\frac{h + \tau \pi_4}{\pi_3 \pi_4} \right) \frac{(\alpha M + \rho(1-\psi)H)}{N} - 1 \right], \\
&= \beta(\tau C + D) \left[\beta(1-\omega) \left\{ \rho(1-\psi)\frac{\rho \psi \pi_3 \pi_4 + (\beta h + \beta \tau \pi_4)}{\pi_3 \pi_4} \frac{H}{N} + \frac{\alpha \beta(h + \tau \pi_4)}{\pi_3 \pi_4} \frac{M}{N} \right\} - 1 \right].
\end{aligned}
$$

At the divorce free equilibrium point,

$$\frac{H}{N} = \frac{\Lambda k \mu (1-r)}{\pi_2 \mu + k(1-r) + kr\pi_2}, \quad \frac{M}{N} = \frac{kr\pi_2 \Lambda}{\pi_2 \mu + k(1-r) + kr\pi_2},$$

hence further simplification results to

$$K'(t) = \beta(\tau C + D) \left\{ \frac{\beta k(1-\omega)}{\pi_3 \pi_4} \left(\frac{\left(\alpha r \pi_2 + \rho \mu(1-\psi)(1-r) \right)(\tau \pi_4 + h) + \rho \psi \mu(1-r)}{\pi_2 \mu + k(1-r) + kr\pi_2} \right) - 1 \right\}.$$

Therefore $K'(t) = \beta(\tau C + D)(R_{eff} - 1)$ and $K'(t) \leq 0$ if $R_{eff} \leq 1$. Hence K is a Lyapunov function in Ω and it follows from LaSalle's invariance principle [4] that every solution of model (1) with initial conditions (2) converges to E_0 as $t \Rightarrow \infty$ i.e. $(C(t), D(t), R(t)) \Rightarrow (0, 0, 0)$ as $t \Rightarrow \infty$. Thus $(S, H, M, C, D, R) \Rightarrow \left(\frac{\Lambda}{\pi_1}, \frac{\Lambda k(1-r)}{\pi_1 \pi_2}, \frac{kr\Lambda}{\mu \pi_1}, 0, 0, 0 \right)$ as $t \Rightarrow \infty$ for $R_{eff} \leq 1$ so that divorce free equilibrium point, E_0 is globally asymptotically stable in Ω if $R_{eff} \leq 1$ for the case where $b = 0$. The significance of this result on the community is that, when there are no record of divorced and separated couples, there will be no restored marriages. Divorce will be reduced to bare minimum in the society if the threshold $R_{eff} < 1$. $\qquad \square$

3.6 Stability of divorce endemic equilibrium, DEE

If $\lambda^* > 0$ and $R_{eff} > 1$, then divorce will persist in the community which is a necessary condition for divorce endemic equilibrium point.

Theorem 3.4. E_1 of model (1) is locally asymptotically stable if $R_{eff} > 1$.

Proof. Linearizing model (1) around the endemic equilibrium, E_1 gives the following Jacobian

$$
J_{E_1} = \begin{pmatrix}
-\pi_1 & 0 & 0 & 0 & b & 0 \\
k(1-r) & -(\rho\lambda^* + \pi_2) & 0 & -\rho\beta\tau(1-\omega)\frac{H^*}{N^*} & -\rho\beta(1-\omega)\frac{H^*}{N^*} & 0 \\
kr & 0 & -(\alpha\lambda^* + \mu) & -\alpha\beta\tau(1-\omega)\frac{M^*}{N^*} & -\alpha\beta(1-\omega)\frac{M^*}{N^*} & 0 \\
0 & \rho(1-\psi)\lambda^* & \alpha\lambda^* & -\pi_3 + \rho\tau(1-\psi)\frac{H^*}{N^*} & \rho(1-\psi)\frac{H^*}{N^*} & 0 \\
0 & \rho\psi\lambda^* & 0 & \rho\beta\tau(1-\omega)\frac{H^*}{N^*} & -\pi_4 + \rho\beta(1-\omega)\frac{H^*}{N^*} & 0 \\
0 & 0 & 0 & \delta & \gamma & -\mu
\end{pmatrix}.
$$

The characteristic polynomial of J_{E_1} written as $|J_{E_1} - wI| = 0$, where w is the eigenvalue and I is a 6×6 identity matrix. Observing J_{E_1}, we can easily see that $w = -\mu$ is an eigenvalue and the remaining eigenvalues are with

$$
\bar{J}_{E_1} = \begin{pmatrix}
-\pi_1 & 0 & 0 & 0 & b \\
k(1-r) & -(\rho\lambda^* + \pi_2) & 0 & -\rho\beta\tau(1-\omega)\frac{H^*}{N^*} & -\rho\beta(1-\omega)\frac{H^*}{N^*} \\
kr & 0 & -(\alpha\lambda^* + \mu) & -\alpha\beta\tau(1-\omega)\frac{M^*}{N^*} & -\alpha\beta(1-\omega)\frac{M^*}{N^*} \\
0 & \rho(1-\psi)\lambda^* & \alpha\lambda^* & -\pi_3 + \rho\tau(1-\psi)\frac{H^*}{N^*} & \rho(1-\psi)\frac{H^*}{N^*} \\
0 & \rho\psi\lambda^* & 0 & \rho\beta\tau(1-\omega)\frac{H^*}{N^*} & -\pi_4 + \rho\beta(1-\omega)\frac{H^*}{N^*} \\
0 & 0 & 0 & \delta & \gamma
\end{pmatrix},
$$

with the corresponding characteristic polynomial written as

$$
\alpha_1 w^5 + \alpha_2 w^4 + \alpha_3 w^3 + \alpha_4 w^2 + \alpha_5 w + \alpha_6 = 0,
$$

where

$$
\begin{aligned}
\alpha_1 &= 1, \\
\alpha_2 &= \pi_1 + A + B + C + \pi_3\tau P + \pi_4, \\
\alpha_3 &= B + AC + \pi_4 + \pi_1(1 + \pi_3\tau P + B) + \pi_3\tau P, \\
\alpha_4 &= \pi_1\{AC + \tau((\pi_3\pi_4 + B)P - \alpha\lambda^* K)\} + (A+C)(\pi_1\pi_4 + B(\pi_1 - \tau P) + \pi_3\tau(\pi_1 + \pi_4)P) \\
&\quad + AC(\pi_3\tau P + \pi_4 + B)B(R_1\tau(C + \pi_4) + R_2 C) + K(bR_2(1-r) - \alpha\tau\lambda^*(A + \pi_4)), \\
\alpha_5 &= \pi_1 AC(\pi_3\tau P + \pi_4 + B) + \tau\Big[\pi_1 P(A+C)(\pi_3\pi_4 - B) + BR_1(\pi_1(C + \pi_4) + bK(1-r)) \\
&\quad + \pi_4 A(\pi_3 PC - \alpha k\gamma^*) - \alpha K\pi_1\lambda^*(A + \pi_4)\Big] + BC(\pi_1 R_2 - \tau(R_2 P(1 + \pi_3) + AP + R_1\pi_4)), \\
\alpha_6 &= \tau\Big[\pi_1\pi_4(\pi_3 ACP + BCR_1 - \alpha AK\lambda^*) + BC(\pi_1 P(A + R_2(1 + \pi_3)) + bKR_1(1-r)) \\
&\quad + bK(\pi_3 MCP(1-r) + \alpha r AB\lambda^*)\Big],
\end{aligned}
$$

where

$$
A = \rho\lambda^* + \pi_2, \quad B = \rho\beta(1-\omega)\frac{H^*}{N^*}, \quad C = \alpha\lambda^* + \mu, \quad K = \rho(1-\psi)\frac{M^*}{N^*},
$$
$$
R_1 = \rho(1-\psi)\lambda^*, \quad R_2 = \rho\psi\lambda^* \quad \text{and} \quad P = \rho(1-\psi)\frac{H^*}{N^*}.
$$

By Routh - Hurwitz stability criterion [17] to obtain roots that are real and negative or has negative real part, the necessary conditions are

$$
\begin{aligned}
&\alpha_i > 0, i = 1, 2, \cdots, 6; \quad \alpha_2\alpha_3\alpha_4 > \alpha_4^2 + \alpha_2^2\alpha_5, \quad \text{and} \\
&(\alpha_2\alpha_5 - \alpha_6)(\alpha_2\alpha_3\alpha_4 - \alpha_4^2 - \alpha_2^2\alpha_5) > \alpha_6(\alpha_2\alpha_3 - \alpha_4)^2 + \alpha_2\alpha_6^2.
\end{aligned} \tag{6}
$$

Observe that $\lambda^* > 0, 0 < \frac{H^*}{N^*} < 1, 0 < \frac{M^*}{N^*} < 1$ and by effect $R_{eff} > 1$. Which also shows that $A, B, C, K, R_1, R_2,$ and P are all greater than zero and in addition (6) is satisfied. Therefore, DEE is locally asymptotically stable for $\lambda^* > 0$ and $R_{eff} > 1$. $\qquad\square$

Theorem 3.5. *The model* (1) *has an divorce endemic equilibrium* (DEE) E_1, *that is globally asymptotically stable whenever* $R_{eff} > 1$.

Proof. Let the state variables for DEE be $X = (S, H, M, C, D, R)$, we define a Dulac's function

$F = \frac{1}{SD}$. Then,

$$
\begin{aligned}
F\frac{dS}{dt} &= \frac{\Lambda}{SR} + \frac{b}{S} - \frac{\pi_1}{D}, \\
F\frac{dH}{dt} &= \frac{k(1-r)}{D} - \frac{\rho\beta\tau(1-\omega)}{SDN}CH - \frac{\rho\beta(1-\omega)}{SN}H - \frac{\pi_2}{SD}H, \\
F\frac{dM}{dt} &= \frac{kr}{D} - \frac{\alpha\beta\tau(1-\omega)}{SDN}CM - \frac{\rho\beta(1-\omega)}{SN}M - \frac{\mu}{SD}, \\
F\frac{dC}{dt} &= \frac{\alpha\beta\tau(1-\omega)}{SDN}CM + \frac{\alpha\rho\beta(1-\omega)}{SN}M + \frac{\rho\beta\tau(1-\psi)(1-\omega)}{SDN}CH + \frac{\rho\beta(1-\psi)(1-\omega)}{SN}M - \frac{\pi_3}{SD}C, \\
F\frac{dD}{dt} &= \frac{\rho\psi\beta\tau(1-\omega)}{SDN}CH + \frac{\rho\psi\beta(1-\omega)}{SN}H + \frac{h}{SD}C - \frac{\pi_4}{S}, \\
F\frac{dR}{dt} &= \frac{\gamma}{S} + \frac{\delta}{SD}C - \frac{\mu}{SD}R.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{dF(X)}{dt} &= \frac{\partial}{\partial S}\left(F\frac{dS}{dt}\right) + \frac{\partial}{\partial H}\left(F\frac{dH}{dt}\right) + \frac{\partial}{\partial M}\left(F\frac{dM}{dt}\right) + \frac{\partial}{\partial C}\left(F\frac{dC}{dt}\right) + \frac{\partial}{\partial D}\left(F\frac{dD}{dt}\right) \\
&\quad + \frac{\partial}{\partial R}\left(F\frac{dR}{dt}\right), \\
&= \frac{\partial}{\partial S}\left(\frac{\Lambda}{SR} + \frac{b}{S} - \frac{\pi_1}{D}\right) + \frac{\partial}{\partial H}\left(\frac{k(1-r)}{D} - \frac{\rho\beta\tau(1-\omega)}{SDN}CH - \frac{\rho\beta(1-\omega)}{SN}H - \frac{\pi_2}{SD}H\right) \\
&\quad + \frac{\partial}{\partial M}\left(\frac{kr}{D} - \frac{\alpha\beta\tau(1-\omega)}{SDN}CM - \frac{\rho\beta(1-\omega)}{SN}M - \frac{\mu}{SD}\right) + \frac{\partial}{\partial C}\left[\frac{\alpha\beta\tau(1-\omega)}{SDN}CM\right. \\
&\quad \left. + \frac{\rho\beta(1-\omega)}{SN}M + \frac{\rho\beta\tau(1-\psi)(1-\omega)}{SDN}CH + \frac{\rho\beta(1-\psi)(1-\omega)}{SN}M - \frac{\pi_3}{SD}C\right] \\
&\quad + \frac{\partial}{\partial D}\left(\frac{\rho\psi\beta\tau(1-\omega)}{SDN}CH + \frac{\rho\psi\beta(1-\omega)}{SN}H + \frac{h}{SD}C - \frac{\pi_4}{S}\right) \\
&\quad + \frac{\partial}{\partial D}\left(\frac{\gamma}{S} + \frac{\delta}{SD}C - \frac{\mu}{SD}R\right), \\
&= -\left[\frac{\Lambda}{S^2R} + \frac{b}{S^2} + \frac{h}{SD^2}C + \frac{\beta\tau(1-\omega)\left((\rho+\alpha)C - \alpha M - \rho(1-\psi)H\right)}{SDN}\right. \\
&\quad \left. + \frac{\rho\beta(1-\omega)(1+\alpha)}{SN} + \frac{\pi_2 + \pi_3 + 2\mu}{SD}\right], \\
&\leq 0.
\end{aligned}
$$

Therefore by Bendixson - Dulac criterion, the system (1) has no periodic orbit i.e. there is no existence of a homoclinic orbit. Hence all solutions of model (1) tend to E_1 for $t \in [0, \infty)$ whenever $R_{eff} > 1$. The significance of the non - existence of periodic orbit means that there are fluctuations in the number of divorced and separated couple in the society at any given time which makes it fiendish to pin - mark resources for divorce control. □

3.7 Bifurcation analysis

With regards to the complexities of the model (1) which has to do with human psychology and sociology we will make use of theorem 4.1 in [6] and center manifold theory to study the bifurcation of the solution manifold. Let the coefficients that represent dynamics on the centre manifold be a_1 and a_2. If $a_1 > 0$ and $a_2 > 0$ we have a backward bifurcation when the bifurcation parameter $\beta^* = 0$, but if $a_1 < 0$ and $a_2 > 0$ it indicates the occurrence of forward bifurcation. Let

the bifurcation parameter be $\beta^* = \beta$ with a critical value obtained when $R_{eff} = 1$, hence,

$$\beta^* = \beta = \frac{\pi_3 \pi_4 \Big(\pi_2(\mu + kr) + k\mu(1 - r) \Big)}{k(1 - \omega)\Big(\big(\alpha r \pi_2 + \rho\mu(1 - \psi)(1 - r) \big)(\tau\pi_4 + h) + \rho\mu\psi(1 - r) \Big)}.$$

Setting $S = x_1$, $H = x_2$, $M = x_3$, $C = x_4$, $D = x_5$, $R = x_6$; then system (1) becomes

$$\left.\begin{aligned}
f_1 \equiv x_1' &= \Lambda + bx_5 - \pi_1 x_1, \\
f_2 \equiv x_2' &= k(1 - r)x_1 - \frac{\rho\beta(1 - \omega)(\tau x_4 + x_5)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}x_2 - \pi_2 x_2, \\
f_3 \equiv x_3' &= krx_1 - \frac{\alpha\beta(1 - \omega)(\tau x_4 + x_5)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}x_3 - \mu x_3, \\
f_4 \equiv x_4' &= \frac{\alpha\beta(1 - \omega)(\tau x_4 + x_5)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}x_3 + \frac{\rho\beta(1 - \omega)(1 - \psi)(\tau x_4 + x_5)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}x_2 - \pi_3 x_4, \\
f_5 \equiv x_5' &= \frac{\rho\beta\psi(1 - \omega)(\tau x_4 + x_5)}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}x_2 + hx_4 - \pi_4 x_5, \\
f_6 \equiv x_6' &= \gamma x_5 + \delta x_4 - \mu x_6.
\end{aligned}\right\} \quad (7)$$
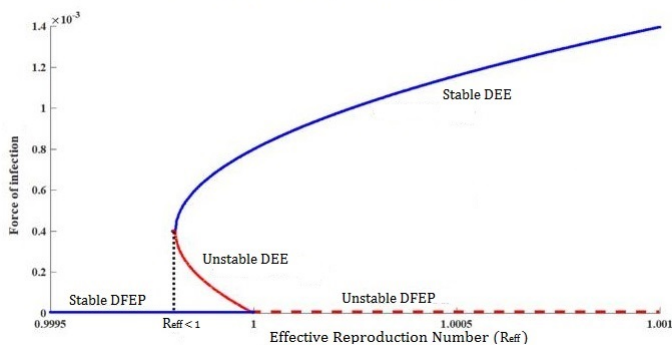
Figure 2: The occurrence of backward bifurcation for $R_{eff} \leq 1$.

The linearization of system (6) around E_0 evaluated at β^* gives

$$J_{E_0} = \begin{pmatrix}
-\pi_1 & 0 & 0 & 0 & b & 0 \\
k(1 - r) & -\pi_2 & 0 & \tau L & L & 0 \\
kr & 0 & -\mu & \tau L' & L' & 0 \\
0 & 0 & 0 & \tau A - \pi_3 & A & 0 \\
0 & 0 & 0 & \tau\psi L + h & \psi L - \pi_4 & 0 \\
0 & 0 & 0 & \delta & \gamma & -\mu
\end{pmatrix},$$

where $L = \dfrac{k\rho\mu\beta^*(1 - \omega)(1 - r)}{\pi_2(\mu + kr) + k\mu(1 - r)}$, $L' = \dfrac{\alpha\pi_2\mu\beta^*(1 - \omega)}{\pi_2(\mu + kr) + k\mu(1 - r)}$ and

$A = \dfrac{\beta^* k(1 - \omega)\Big(\alpha r \pi_2 + \rho\mu(1 - \psi)(1 - r) \Big)}{\pi_2(\mu + kr) + k\mu(1 - r)}$. The characteristic equation $|J_{E_0} - \lambda I| = 0$, taking into

account that $\beta = \beta^*$ has a simple zero eigenvalue and other eigenvalues have negative real sign, hence E_0 is a non - hyperbolic equilibrium point. To compute a right eigenvector, w, we examine $J_{E_0} w = 0$. Assuming $w = (w_1, w_2, w_3, w_4, w_5, w_6)^T$, we get

$$
\begin{aligned}
w_1 &= \frac{b}{\pi_1} w_5, \\
w_2 &= \frac{(kb(1-r) + L\pi 1)(\pi_3 - \tau A) + \tau \pi_1 L}{\pi_1 \pi_2 (\pi_3 - \tau A)} w_5, \\
w_3 &= \frac{\tau \pi_1 A L' + (krb + \pi_1 L')(\pi_3 - \tau A)}{\mu \pi_1 (\pi_3 - \tau A)} w_5, \\
w_4 &= \frac{A}{\pi_3 - \tau A} w_5, \\
w_5 &> 0, \\
w_6 &= \frac{\delta(\psi L - \pi_4) + \gamma(\tau \psi L + h)}{\mu(\tau \psi L + h)} w_5.
\end{aligned}
$$

Next, we compute the left eigenvector, v by solving $J_{E_0} v = 0$. Assuming $v = (v_1, v_2, v_3, v_4, v_5, v_6)$, we obtain $v_1 = v_2 = v_3 = v_6 = 0$, $v_4 = \frac{\tau \psi L + h}{\pi_3 - \tau A}$, $v_5 = v_5 > 0$. To compute the coefficients of dynamics of the system (6) we make use of theorem 4.1 in [6]:

$$
a_1 = \sum_{k,i,j=1}^{6} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_1 \partial x_j}(E_0, \beta^*), \quad a_2 = \sum_{k,i=1}^{6} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(E_0, \beta^*).
$$

Since $v_1 = v_2 = v_3 = v_6 = 0$, we do not need the derivatives of f_1, f_2, f_3, and f_6. The second derivatives of f_4 and f_5 that are nonzero are

$$
\begin{aligned}
\frac{\partial^2 f_4}{\partial x_2 \partial x_4} &= \rho \beta (1 - \psi)(1 - \omega) \left(\frac{\mu(\pi_1 \pi_2 - \Lambda k(1-r))}{\pi_2(\mu + kr) + k\mu(1-r)} \right) = \frac{\partial^2 f_4}{\partial x_4 \partial x_2}, \\
\frac{\partial^2 f_4}{\partial x_2 \partial x_5} &= \frac{\partial^2 f_4}{\partial x_5 \partial x_2} = \rho \beta^* (1 - \psi)(1 - \omega) \left(\frac{\mu(\pi_1 \pi_2 - \Lambda k(1-r))}{\pi_2(\mu + kr) + k\mu(1-r)} \right), \\
\frac{\partial^2 f_5}{\partial x_2 \partial x_4} &= \frac{\partial^2 f_5}{\partial x_4 \partial x_2} = \rho \beta^* \psi \tau (1 - \omega) \left(\frac{\mu(\pi_1 \pi_2 - \Lambda k(1-r))}{\pi_2(\mu + kr) + k\mu(1-r)} \right), \\
\frac{\partial^2 f_5}{\partial x_2 \partial x_5} &= \frac{\partial^2 f_5}{\partial x_5 \partial x_2} = \rho \beta^* \psi (1 - \omega) \left(\frac{\mu(\pi_1 \pi_2 - \Lambda k(1-r))}{\pi_2(\mu + kr) + k\mu(1-r)} \right), \\
\frac{\partial^2 f_4}{\partial x_4^2} &= -2\alpha \beta^* \tau (1 - \omega) \left(\frac{r\pi_1 \pi_2 - k\mu(1-\psi)(1-r)}{\pi_2(\mu + kr) + k\mu(1-r)} \right), \\
\frac{\partial^2 f_4}{\partial x_4 \partial \beta} &= \frac{k\tau(1 - \omega)\big(\alpha r + \rho(1 - \psi)(1 - r)\big)}{\pi_2(\mu + kr) + k\mu(1-r)}, \\
\frac{\partial^2 f_5}{\partial x_4 \partial \beta} &= \frac{k\rho \psi \tau \mu (1 - \omega)(1 - r)}{\pi_2(\mu + kr) + k\mu(1-r)}, \qquad \frac{\partial^2 f_4}{\partial x_5 \partial \beta} = \frac{k(1 - \omega)(\alpha r + \rho(1 - \psi)(1 - r))}{\pi_2(\mu + kr) + k\mu(1-r)}, \\
\frac{\partial^2 f_5}{\partial x_5 \partial \beta} &= \frac{k\rho \psi \mu (1 - \omega)(1 - r)}{\pi_2(\mu + kr) + k\mu(1-r)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
a_1 &= v_4\left[2\left(w_2w_4\frac{\partial^2 f_4}{\partial x_4\partial x_2}+w_2w_5\frac{\partial^2 f_4}{\partial x_2\partial x_5}\right)+w_4^2\frac{\partial^2 f_4}{\partial x_4^2}\right]+2v_5\left[w_2w_4\frac{\partial^2 f_5}{\partial x_2\partial x_4}+w_2w_5\frac{\partial^2 f_5}{\partial x_2\partial x_5}\right],\\
&= \frac{2\rho\beta\mu(1-\omega)\Big(\pi_1\pi_2-\Lambda k(1-r)\Big)}{\pi_2(\mu+kr)+k\mu(1-r)}\Big[(\tau w_2w_4+w_2w_5)(v_4(1-\psi)+\psi v_5)\\
&\quad +\frac{v_4(1-\psi)\Big(r\pi_1\pi_2-k(1-\psi)(1-r)\Big)}{2\Big(\pi_1\pi_2-\Lambda k(1-r)\Big)}\Big]>0.\\
a_2 &= \frac{k(\tau w_4+w_5)(\omega)}{\pi_2(\mu+kr)+k\mu(1-r)}\Big[v_4(\alpha r+\rho(1-\psi)(1-r))+v_5\rho\mu\psi(1-r)\Big]>0.
\end{aligned}
$$

From the above $a_1>0$ and $a_2>0$ at $\beta=\beta^*$. By theorem 4.1 in [6], model (1) exhibits backward bifurcation at $R_{eff}=1$. The associated bifurcation diagram is depicted in Figure 2, and shows that the local asymptotically stables, DEE and DFEP are depended on initial sizes of the sub-population which explains the coexistence of the stable states. The implication of this result is that the classical prerequisite that makes $R_{eff}<1$ are necessary condition but not sufficient for the elimination of divorce related cases in the society. Therefore, stable divorce endemic equilibrium co - exists with divorce free equilibrium when $R_{eff}\leq 1$ because divorce can be sensed as an issue of behavioural responses to negative affects, high expectations, inordinate communication strategies, lack of trust and intimacy.

Succinctly, the mathematical analysis in this section shows that having previously divorced individuals becoming susceptible to divorce again will make it difficult to have divorce eradicated from the population.

4 Sensitivity Analysis of R_{eff}

In model (1), must of the essential static quantities depend on the parameters of the differential equation. Effective reproduction number is used to measure the sensitivity of the model parameters by computation of the normalized forward sensitivity indices of R_{eff}. This gives the percentage of influence each parameter has on divorce transmission dynamics and prevalence. The sensitivity elasticity of quantity R_{eff} with respect to the parameter β_i is given by

$$
\xi_{\beta_i}^{R_{eff}}=\frac{\partial R_{eff}}{\partial\beta_i}\times\frac{\beta_i}{R_{eff}}.
$$

Therefore,

$$
\begin{aligned}
\xi_\beta^{R_{eff}} &= 1,\quad \xi_k^{R_{eff}}=\frac{\pi_2\mu}{\pi_2\mu+k(1+r(1+\pi_2))},\\
\xi_\omega^{R_{eff}} &= -\frac{k\beta}{\pi_3\pi_4}\left[\frac{(\alpha r\pi_2+\rho\mu(1-\psi)(1-r))(\tau\pi_4+h)+\rho\psi\mu(1-r)}{\pi_2\mu+k(1+r(1+\pi_2))}\right],\\
\xi_\psi^{R_{eff}} &= -\frac{\rho\mu(1-r)((\tau+h)-1)}{\pi_3\pi_4(\pi_2\mu+k(1+r(1+\pi_2)))},\quad \xi_\tau^{R_{eff}}=\frac{\beta k\pi_4(1-\omega)(\alpha r\pi_2+\rho\mu(1-\psi)(1-r))}{\pi_3\pi_4(\pi_2\mu+k(1+r(1+\pi_2)))},\\
\xi_\rho^{R_{eff}} &= \frac{\beta k\mu(1-\omega)(1-r)((1-\psi)(\tau\pi_4+h)-\psi)}{\pi_3\pi_4(\pi_2\mu+k(1+r(1+\pi_2)))},\quad \xi_\alpha^{R_{eff}}=\frac{\beta kr\pi_2(1-\omega)}{\pi_3\pi_4(\pi_2\mu+k(1+r(1+\pi_2)))},\\
\xi_r^{R_{eff}} &= -\frac{\rho\mu r((1-\psi)(\tau\pi_4+h)+\psi)}{(\alpha r\pi_2+\rho\mu(1-\psi)(1-r))+\rho\psi\mu(1-r)}.
\end{aligned}
$$

Parameters with positive indices have an enormous effect on the transmission and prevalence of divorce in the society i.e. increase in such parameter enhances the multiplication of divorce in the society that leads to mayhem in the existence of a peaceful and productive society.

On the other hand, the parameters that have negative indices have minimizing effects on the burden of divorce in the society as their values are increased; hence they are used in the control of divorce within the community and in turn bring sanctity to the community. Using contour plots we investigate the effect of the parameters in model (1) on the effective reproduction number R_{eff}, which we used as a threshold of divorce in the society.

In Figure 3, the contours show that any habit that will increase the divorce transmission rate by either physical or virtual contact will aid in the proliferation of divorce cases; and when the singles prepare and equip themselves with knowledge of marriage, family dynamics, skills, and personal awareness, R_{eff} will be reduced thereby reducing the burden of divorce in the society.

Furthermore, in Figure 4, we can see that increasing the level of tolerance of negative affects of marriage partners helps greatly in reducing R_{eff}, which in turn reduces the prevalence of divorce in the society. Proper preparation for marriage in all its tendencies helps in developing tolerance of negative affects when faced with one during the marriage; hence the combination of r and ω brings the divorce rate down drastically.
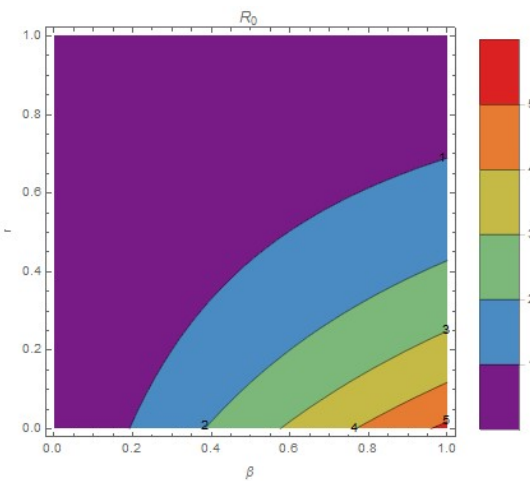
Figure 3: Contour plots of R_0 versus transmission rate and marriage preparedness of singles who are ready for marriage.
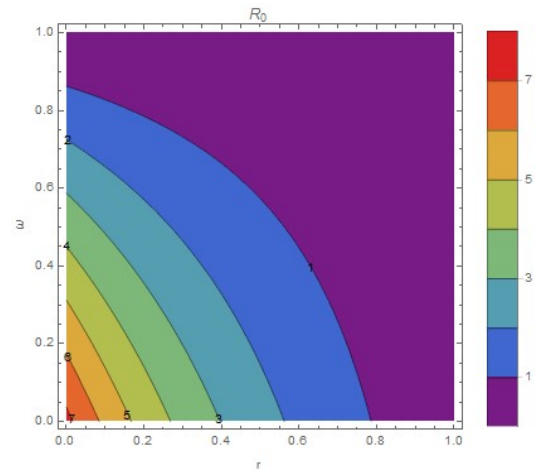
Figure 4: Contour plots of R_0 versus marriage preparedness and the tolerance level of couples on negative affects exhibited by their partner.

The contour in Figure 5 shows that when the level of dysfunctional and escalated negative affect is high in a marriage, it is bound to witness separation which may result in divorce. On the other - hand, being conversant with family dynamics before venturing into marriage will reduce the level of dysfunctions in marriages which will result in partial containment of divorce cases. Figure 6 shows that the opposite of knowledge of family dynamics and self-discovery will lead to an increment in transmission dynamics and prevalence of divorce in society.
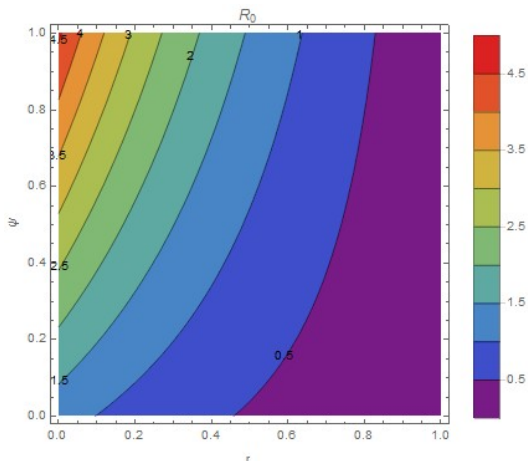
Figure 5: Contour plots of R_0 versus marriage preparedness and level of dysfunctional which is as a result of escalated negative affects.
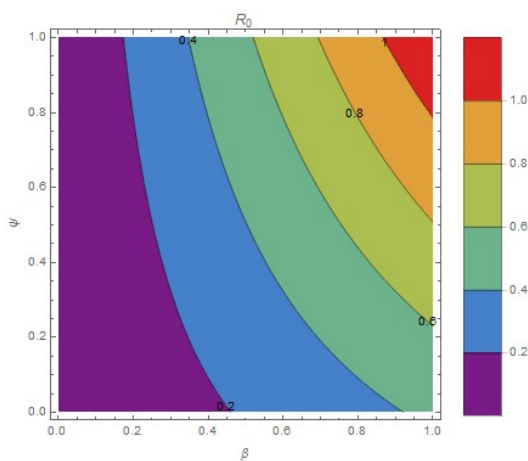

Figure 6: Contour plots of R_0 versus divorce transmission rate and level of dysfunctional which is as a result of escalated negative affects.
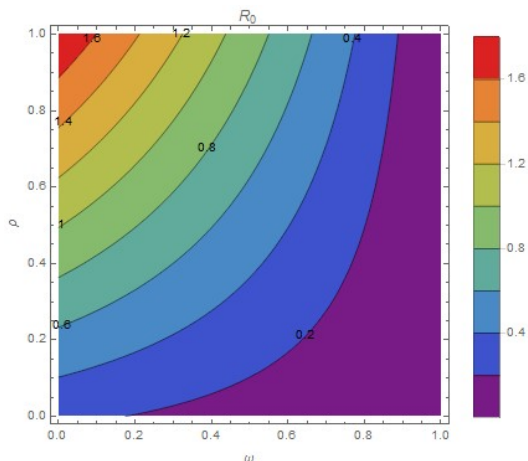

Figure 7: Contour plots of R_0 versus tolerance to negative affects and proportion of unstable marriage that leads to either separation or divorce.
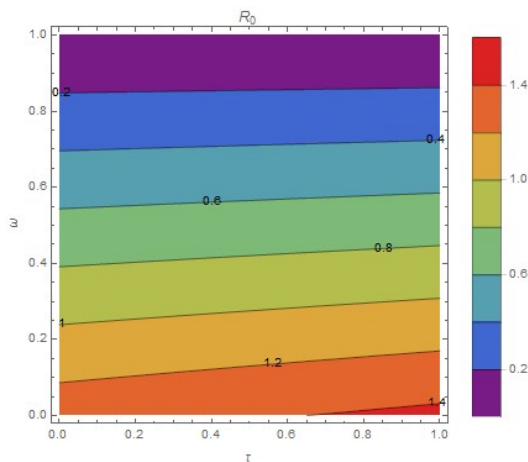

Figure 8: Contour plots of R_0 versus modification factor of separated couple and tolerance level of negative affects.

In Figure 7, it is observed that the increase in tolerance level exhibited in any marriage helps in the reduction of the proportion of troubled couples that may opt for separation or divorce. Couples over time head to divorce because they fail to work out a common influence pattern; in addition, the majority argue about differences in how to express emotions, closeness, and distances. These purposeful learning of traits that can douse tension in marriages will help in modification of the divorce ideology being considered by the separated class as shown in Figure 8.

5   Numerical Simulation

In this section we estimate the parameters of model (1) based on data available in Nigeria and secondary data for numerical simulation performed to support the analytical results. The values of parameters used in the simulation is shown in Table 2 with some estimated, assumed,

calculated and obtained from literature, initial conditions used is follows: $S(0) = 20000, H(0) = 3000, M(0) = 4000, C(0) = 1500, D(0) = 500$ and $R(0) = 100$.

Table 2: Biological, sociological and psychological interpretation of parameters involved in the model system 1.

Parameters	Value(s)	Source
Λ	0.4	[8]
b	0.001	Assumed
k	0.501	Estimated
μ	0.001495	Calculated [19]
r	0.7	Extimated, $(0, 1]$
ρ	0.72	Assumed
α	0.086	[21]
ψ	0.74	Estimated, $(0, 1]$
h	0.24	[13]
δ	0.3	[9]
γ	0.1	[13]
ω	0.49	Estimated $(0, 1]$
β	0.71	[21]
τ	0.241	[13]
d_1	0.0021	[9]
d_2	0.00182	Assumed

The community has to work enormously on the health of families which are an integral component of any society. As married couples live together, a marital disorder in the form of separation or divorce may always be experienced among them because of individual differences, religion, race, family practices, and other inevitable psychological and sociological traits.

In Figure 9, it is observed that when the divorce transmission parameter is boosted, the divorced population is on the increase. This increment is not perpetual, it will start to dwindle gradually since divorce/separation is psychological or otherwise. The divorcees will start to remarry and others will reconcile with their spouses cautioned by reconciliation measures marshalled out by concerned individuals, hence leading to depopulation of the divorced population and population of the restored marriages as shown in Figure 10.
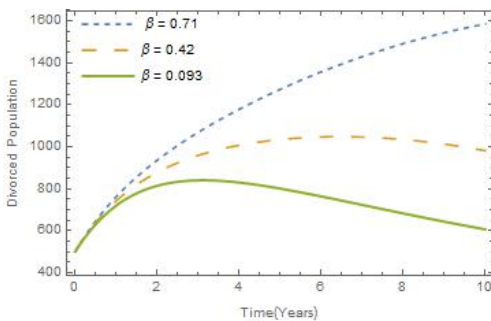
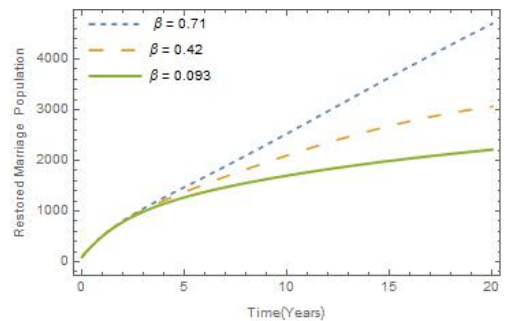Figure 9: Effect of β on the divorced population.

Figure 10: Effect of β on the restored marriage population.

The effect of these reconciliation measures reduces the population of the separated couples because some of them reconsider and discover themselves on the track which fast tracks the effort of

reconciliation as shown in Figure 11. Therefore, separated or divorced couples will remain broken without an adequate and consistent reconciliation process. For the unstable marriage population as in Figure 12 when β is increased, it has a deplorable effect on the population as a result of the mass exodus to divorced or separated compartments.
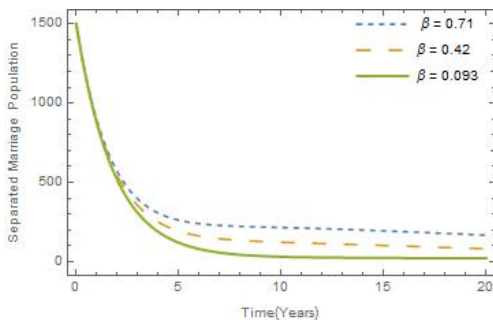

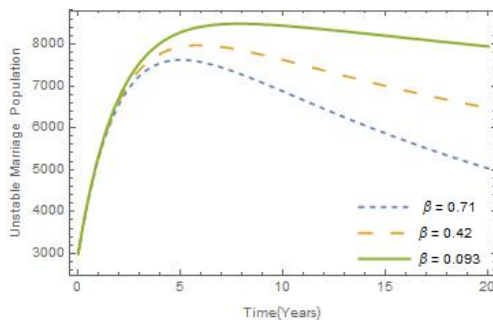Figure 11: Effect of β on the separated marriage population.

Figure 12: Effect of β on the unstable marriage population.

When couples intentionally develop, learn and practice tolerance of negative affect exhibited by their spouse, there will be peace in the society hence population of divorced will be reduced drastically as shown in Figure 13. This co-existence strategy affects both stable and unstable marriages because new trait that is inherent or acquired will be displayed daily hence the need to contain them is needful. Figures 14 and 15 show that the population of both stable and unstable marriages is increased when a conscientious effort is applied to upgrading the ω level. In addition, the population of restored marriages and separated marriages are reduced, but not drastically. Over time there is always a need to unlearn, learn and relearn to keep track of the complexities of human behavioural, attitudinal, and sociological well-being as shown in Figures 16 and 17 respectively.

Figures 18 and 19 show that if enormous significance is placed on preparedness before entering into marriage, divorce and separation rates shall be reduced, to a manageable extent. This preparedness for marriage should not be static but dynamic because of the dynamism of human psychological and sociological variations. Peradventure divorce or separation occurs, and the reconciliation process is faster with a spouse who worked on self before entering into a marriage as shown in Figure 20. If a considerable number of singles who are ready to get married devote their time intentionally to learning the family dynamics that include: communication of all types - words, expressions, attitudes, expectations, emotions, thoughts, and forces outside the family, such as history, context, e.t.c. [12]; the majority of marriages will start as stable marriages, hence depopulating unstable marriages as shown in Figures 21 and 22.
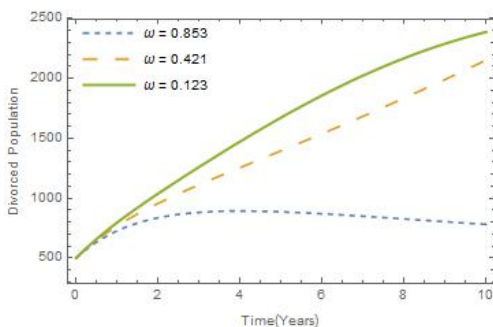

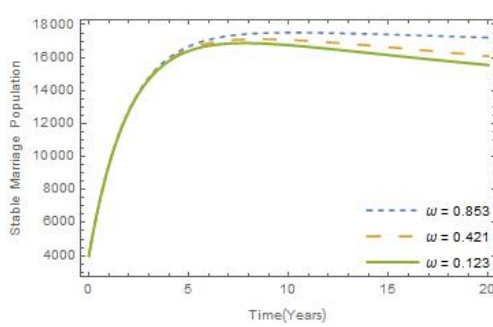Figure 13: Effect of ω on the dissolved marriage population.

Figure 14: Effect of ω on the stable marriage population.
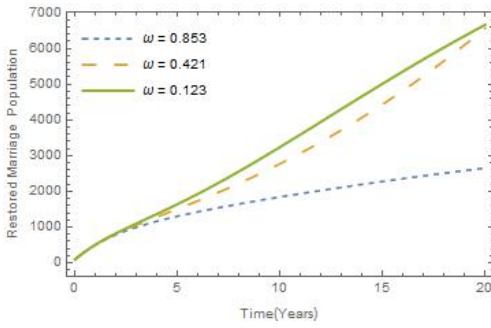
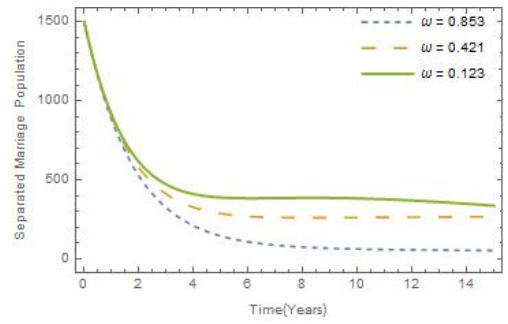Figure 15: Effect of ω on the restored marriage population.

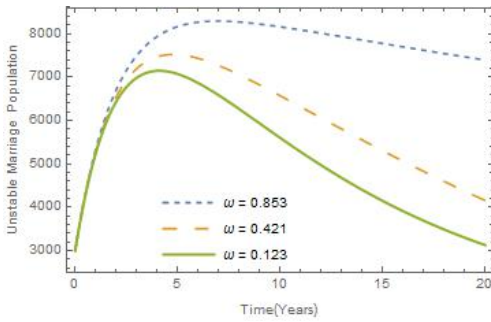Figure 16: Effect of ω on the separated marriage population.

Figure 17: Effect of β on the unstable marriage population.
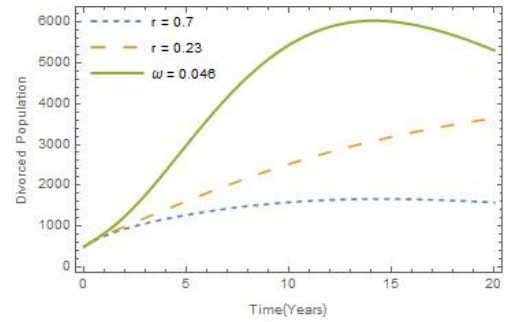

Figure 18: Effect of r on the divorced marriage population.
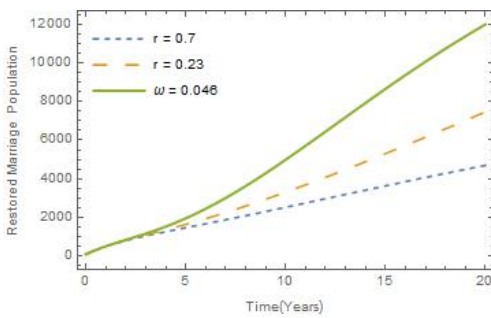

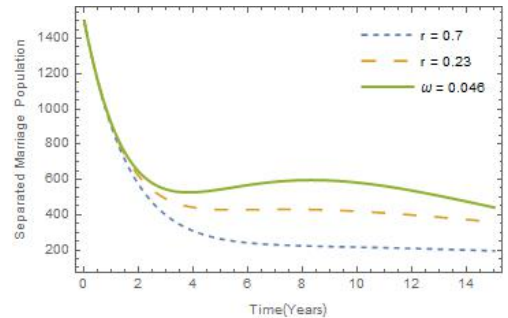Figure 19: Effect of r on the restored marriage population.

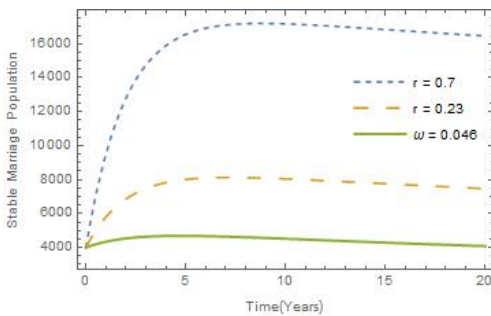Figure 20: Effect of r on the separated marriage population.

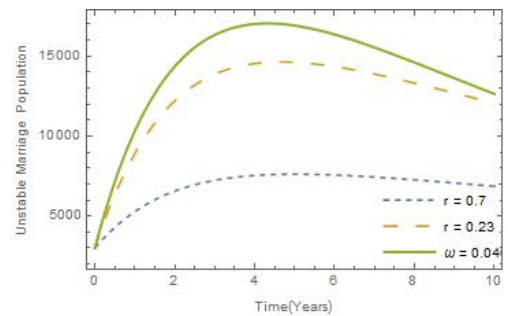Figure 21: Effect of r on the stable marriage population.

Figure 22: Effect of r on the unstable marriage population.

6 Conclusion

A deterministic non-linear ordinary differential system of equations was used to study the effect of pre-marriage preparedness with interpersonal tolerance among the married on the dynamics of divorce/separation using a mathematical model as a tool. The model was propounded and analyzed, considering divorce as an epidemic in Nigeria. A feasible region was obtained in the model analysis and its epidemiological meaningfulness was established. The divorce-free equilibrium and divorce endemic equilibrium were obtained and their local and global stability were studied. The bifurcation analysis of the model was performed and it showed that the model exhibits backward bifurcation at $R_{eff} = 1$. The sensitivity of the model parameters and numerical simulation was performed to expatiate the analytical results. The simulation suggests that pre-marriage preparedness, interpersonal tolerance in marriage, and reconciliation measures help stabilize marriages, prevent cracks, and rectify separation/divorce marriages. By and large, society should be involved in indoctrinating, teaching, and practising the family dynamics to enhance the learning of marriage norms at a tender age. Parents, churches, mosques, and the elderly should endeavour to have preparatory sessions for singles that are getting ready to enter into the institution of marriage. The sessions should cover a variety of substantive areas that include reproductive issues (childbearing, parenthood, infertility, childlessness, adoption), household decisions, life course decisions, health decisions, and institutional decisions.

Conflicts of Interest The authors declare no conflict of interest.

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